

# Enrique Surfaces

Definition:  $X =$  Enrique surface

w/  $q = P_g = 0$      $K_X^{\otimes 2} \cong \mathcal{O}_X$     but  $K_X \not\cong \mathcal{O}_X$

Noether's formula  $\implies \chi_{\text{top}}(X) = 12$

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\therefore \text{Pic}(X) \cong H^2(X, \mathbb{Z})$   
 $D \longmapsto d$

$d \in H^2(X, \mathbb{Z})$ ,  $d^2 = D^2 \equiv D \cdot K_X = 0 \pmod{2}$  (adjunction)

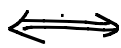
$\tau(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X)) = -8$  (topological index theorem)

Thus, the pairing on  $H^2(X, \mathbb{Z})/\text{tor}$  is unimodular, even,

indefinite of rank 10.  $\implies H^2(X, \mathbb{Z})/\text{tor} \cong H^{\oplus}(-E_8)$

$X$ : variety

$L \in \text{Pic}(X)$  s.t.  $L^{\otimes 2} \cong \mathcal{O}_X$



$\tilde{X} \xrightarrow{\pi} X$  étale double cover  
 $\pi^*L = \mathcal{O}_{\tilde{X}}$

Proposition 1:

- $X =$  Enrique surface,  $K_X \rightsquigarrow \tilde{X}$  is a K3 surface
  - $\tilde{X} =$  K3 surface  $\curvearrowright \sigma$  fixed-point free involution
- $X = \tilde{X}/\sigma$  is an Enrique surface.

pf:  $\left. \begin{aligned} & K_{\tilde{X}} = \pi^*(K_X \otimes_{\text{as } L} K_X) = \mathcal{O}_{\tilde{X}} \\ & \chi(\mathcal{O}_{\tilde{X}}) = 2\chi(\mathcal{O}_X) = 2 \quad \therefore g(\tilde{X}) = 0 \end{aligned} \right\} \Rightarrow \tilde{X} \text{ K3 surface}$

$\tilde{X} \xrightarrow{\pi} X$  étale cover,  $\tilde{X}$  minimal  $\Rightarrow X$  minimal.

$\pi^*K_X \cong K_{\tilde{X}} = \mathcal{O}_{\tilde{X}} \Rightarrow 2K_X = \pi_*\pi^*K_X = 0$  as divisors

$\chi(\mathcal{O}_X) = \frac{1}{2}\chi(\mathcal{O}_{\tilde{X}}) = 1$

$\therefore g(X) = 0, P_g(X) = 0$

ex  $\tilde{X} =$  intersection of 3 generic quadrics in  $\mathbb{P}^5$  of the form

$P_i = Q_i(x_0, x_1, x_2) + Q'_i(x_3, x_4, x_5) = 0, \quad i=1, 2, 3$

K3 surface by adjunction formula

$\sigma: \mathbb{P}^5 \xrightarrow{\hspace{10em}} \mathbb{P}^5$  sending  $\tilde{X}$  to  $\tilde{X}$   
 $(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5)$

$\text{Fix}(\sigma) = \{x_3 = x_4 = x_5 = 0\} \cup \{x_1 = x_2 = x_3 = 0\}$  two 2-planes

$Q_i, Q'_i$  generic. then  $\tilde{X} \cap \text{Fix}(\sigma) = \emptyset$

say  $Q_i$  (or  $Q'_i$ ) has no common intersections on the 2-planes

There exists an elliptic fibration on  $X$ .

$\exists \lambda, \mu, \nu$ , s.t.  $\lambda Q_1 + \mu Q_2 + \nu Q_3, \lambda Q'_1 + \mu Q'_2 + \nu Q'_3$

both degenerate conics.

imposes a deg 3 poly. in  $\lambda, \mu, \nu$

wlog,  $\lambda \neq 0$  replace  $P_1$  by  $\lambda P_1 + \mu P_2 + \nu P_3$

$\rightsquigarrow P_1$  contains a pencil of 3-planes  $(L_t)_{t \in \mathbb{P}^1}$ .

$L_t \cap X = L_t \cap P_2 \cap P_3$  generically smooth elliptic curve.

Proposition 2.  $X =$  Enriques surface  $\cong E$  elliptic curve

Then either ①  $h^0(E) = 1$ ,  $|2E|$  base point free pencil

or ②  $|E|$  base point free pencil w/  
exactly 2 multiple fibres  $E_1, E_2$ .

pf:

$$\begin{aligned}
 & 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(E) \rightarrow \mathcal{O}_E(E) \rightarrow 0 \\
 & \rightsquigarrow 0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, E) \rightarrow H^0(E, \mathcal{O}_E(E)) \rightarrow 0 \\
 & \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, E) \rightarrow H^1(E, \mathcal{O}_E(E)) \rightarrow H^2(X, \mathcal{O}_X) = 0
 \end{aligned}$$

degree 0  $\because E^2 = 0$   
by adjunction

①  $h^0(E, \mathcal{O}_E(E)) = 0$ , then  $h^0(E) = 1$ .

$\rightarrow$

$$\text{RR. } \underbrace{h^0(-E)}_0 - h^1(-E) + h^0(K_X + E) = \chi(\mathcal{O}_X) + \frac{1}{2}(-E) \cdot (-E - K_X) = 1$$

$$\Rightarrow |K_X + E| \neq \emptyset \quad E' \cdot E = 0$$

$E'$

$$\underbrace{h^0(-E-E')}_{0} - h^1(-E-E') + h^0(\underbrace{K_X + E + E'}_{2E}) = \chi(\mathcal{O}_X) + \frac{1}{2}(-E-E') \cdot (-E-E'-K_X) = 1$$

$\because 2K_X \sim 0$

$$\therefore h^0(2E) = 1 + h^1(-E-E')$$

$$0 \rightarrow \mathcal{O}_X(-E-E') \rightarrow \mathcal{O}_X(-E') \rightarrow \mathcal{O}_E(-E') \rightarrow 0$$

$$0 = H^0(X, \mathcal{O}_X(-E')) \rightarrow H^0(E, \mathcal{O}_E(-E')) \xrightarrow{\cong} H^1(X, \mathcal{O}_X(-E-E')) \Rightarrow h^1(-E-E') \geq 1$$

$E$  effective  $\cong$   $\cong$   $\therefore h^0(2E) \geq 2$

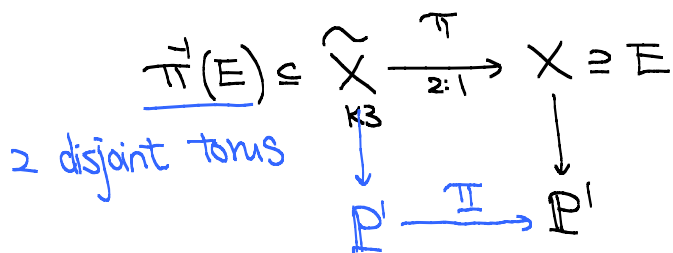
from previous discussion

$$h^0(2E) = 2$$

②  $h^0(E, \mathcal{O}_E(E)) \neq 0$ , then  $\mathcal{O}_E(E) \cong \mathcal{O}_E$  &  $h^0(E) = 2$

adjunction  $\Rightarrow E^2 = 0 \therefore |E|$  has no base points

$X \xrightarrow{|E|} \mathbb{P}^1$  elliptic fibration



Thus,  $(\Pi)$  is a double cover  
Möbius transf.

has two ramification points

$\Rightarrow$  The pre-images are the two multiple fibres  $E_1, E_2$

$$2E_1 \sim E \sim 2E_2$$

$$\tilde{X} \xrightarrow[2:1]{} X \Rightarrow \pi_1(X) \cong \mathbb{Z}_2 \quad \therefore H_1(X, \mathbb{Z}) \cong \mathbb{Z}_2$$

$\pi_1(\tilde{X}) = 0$

universal coefficient theorem

$$0 \rightarrow \text{Ext}^1(H_1(X, \mathbb{Z}), \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \rightarrow \text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \rightarrow 0$$

$$\Rightarrow H^2(X, \mathbb{Z})_{\text{tor}} \cong \mathbb{Z}_2$$

generated by  $K_X$ ,  $2K_X \sim 0$

$$H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\cong} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)$$

$\underset{0}{\parallel}$   $\underset{0}{\parallel}$

$$\Rightarrow K_X \sim E_1 - E_2 \quad 2(E_1 - E_2) \sim 0$$

Proposition 3: Every Enriques surface admits an elliptic fibration.

pf: Let  $X$  be an Enriques surface.  
 $\tilde{X}$  be the double cover K3 surface.

Choose  $d \in H^2(X, \mathbb{Z})_{\text{tor}}$  s.t.  $d^2 = 0$ ,  
 $\cong$   
 $H^2(-E_8)$

$$H^1(X, \mathcal{O}_X) \rightarrow \text{Pic } X \cong H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)$$

$\underset{0}{\parallel}$   $\exists \tilde{D} \mapsto \tilde{d}$   $\underset{0}{\parallel}$

R.R  $\underbrace{h^0(X, D) + h^0(X, K_X - D)}_{\text{exactly one of them nonzero}} \geq \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X) = 1 + \frac{1}{2} D^2 = 1$

Similarly,  $\underbrace{h^0(X, -D) + h^0(X, K_X + D)}_{\text{exactly one of them nonzero}} \geq 1 \quad (K_X + D)^2 = D^2 = 0$

$|K_X \pm D|$  can't be both non-trivial since  $(K_X + D) + (K_X - D) \sim 0$

Therefore, exactly one of  $|D|, |-D|$  is non-empty.

In particular,  $\exists$  effective  $D$  on  $X$  w/  $D^2 = 0$ .

Similar to the K3 case, after finitely many Picard-Lefschetz

reflection, we may assume that  $D.C \geq 0$ ,  $V(-2)$ -curve.

Claim:  $D = mF$ . for some elliptic configuration.

Then the proposition follows from the Proposition 3.