Enrique Surfaces

Definition: X = Enrique surface

w/ $q = P_g = 0$ $K_X^{\otimes 2} \cong \mathcal{O}_X$ but $K_X \not\cong \mathcal{O}_X$

Noether's formula $\Longrightarrow$ $\chi_{top}(X) = 12$

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\therefore$ $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$

$D \longmapsto d$

$d \in H^2(X, \mathbb{Z})$, $d^2 = D^2 \equiv D.K_X = 0 \pmod 2$

adjunction

$\tau(X) = \frac{1}{3}\left(c_1^2(X) - 2c_2(X)\right) = -8$ topological index theorem

Thus, the pairing on $H^2(X, \mathbb{Z})/tor$ is unimodular, even,

indefinite of rank 10. $\Longrightarrow$ $H^2(X, \mathbb{Z})/_{tor} \cong H \oplus (-E_8)$

X: variety

$L \in \mathrm{Pic}(X)$ s.t $L^{\otimes 2} \cong \mathcal{O}_X$ $\Longleftrightarrow$ $\tilde{X} \xrightarrow{\pi} X$ étale double cover

$\pi^* L = \mathcal{O}_{\tilde{X}}$

Proposition 1:

- X = Enrique surface, K_X $\rightsquigarrow$ $\tilde{X}$ is a K3 surface
- $\tilde{X}$ = K3 surface $\circlearrowright$ σ fixed-point free involution

$X = \tilde{X}/\sigma$ is an Enrique surface.

pf: • $K_{\tilde{X}} = \pi^*(K_X \otimes \underbrace{K_X}_{\text{as } L}) = \mathcal{O}_{\tilde{X}}$

$\chi(\mathcal{O}_{\tilde{X}}) = 2\chi(\mathcal{O}_X) = 2 \quad \therefore q(\tilde{X}) = 0$ $\Big\} \Rightarrow \tilde{X}$ K3 surface

• $\tilde{X} \xrightarrow{\pi} X$ étale cover, $\tilde{X}$ minimal $\Longrightarrow X$ minimal.

$\pi^* K_X \cong K_{\tilde{X}} = \mathcal{O}_{\tilde{X}} \Longrightarrow 2K_X = \pi_* \pi^* K_X = 0$ as divisors

$\chi(\mathcal{O}_X) = \frac{1}{2}\chi(\mathcal{O}_{\tilde{X}}) = 1$

$\therefore q(X) = 0, \ P_g(X) = 0$

ex $\tilde{X}$ = intersection of 3 generic quadrics in $\mathbb{P}^5$ of the form

$P_i = Q_i(x_0, x_1, x_2) + Q_i'(x_3, x_4, x_5) = 0 \quad , \quad i = 1, 2, 3$

K3 surface by adjunction formula

$\sigma : \mathbb{P}^5 \longrightarrow \mathbb{P}^5$ sending $\tilde{X}$ to $\tilde{X}$

$(x_0, x_1, x_2, x_3, x_4, x_5) \longmapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5)$

$\mathrm{Fix}(\sigma) = \{x_3 = x_4 = x_5 = 0\} \cup \{x_1 = x_2 = x_3 = 0\}$ two 2-planes

Q_i, Q_i' generic. then $\tilde{X} \cap \mathrm{Fix}(\sigma) = \phi$

say Q_i (or Q_i') has no common intersections on the 2-planes

There exists an elliptic fibration on X.

$\exists \lambda, \mu, \nu$, s.t $\underline{\lambda Q_1 + \mu Q_2 + \nu Q_3}$, $\lambda Q_1' + \mu Q_2' + \nu Q_3'$

both $\underline{\text{degenerate conics}}$.

imposes a deg 3 poly. in λ, μ, ν

WLOG, $\lambda \neq 0$ replace P_1 by $\lambda P_1 + \mu P_2 + \nu P_3$

$\rightsquigarrow$ P_1 contains a pencil of 3-planes $(L_t)_{t \in \mathbb{P}^1}$.

$L_t \cap X = L_t \cap P_2 \cap P_3$ generically smooth elliptic curve.

Proposition 2. X = Enrique surface $\supseteq E$ elliptic curve

Then either ① $h^0(E) = 1$, $|2E|$ base point free pencil

or ② $|E|$ base point free pencil w/ exactly 2 multiple fibres E_1, E_2.

pf:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_E(E) \to 0 \qquad \text{degree } 0 \because E^2 = 0 \text{ by adjunction}$$

$$\rightsquigarrow 0 \to \underset{\mathbb{C}}{\underset{\cong}{H^0(X, \mathcal{O}_X)}} \to H^0(X, E) \to H^0(E, \mathcal{O}_E(E)) \to 0$$

$$\to \cancel{H^1(X, \mathcal{O}_X)}^{\,0} \to H^1(X, E) \to H^1(E, \mathcal{O}_E(E)) \to H^2(X, \mathcal{O}_X) = 0$$

① $\underline{h^0(E, \mathcal{O}_E(E)) = 0}$, then $h^0(E) = 1$.

$\Rightarrow$

RR. $$\underset{0}{\underset{\parallel}{h^0(-E)}} - h^1(-E) + h^0(K_X + E) = \underset{1}{\underset{\parallel}{\chi(\mathcal{O}_X)}} + \underset{0}{\underset{\parallel}{\underline{\tfrac{1}{2}(-E).(-E-K_X)}}} = 1$$

$$\Longrightarrow \quad \underset{E'}{\underset{\Psi}{|K_X + E|}} \neq \phi \qquad E'.E = 0$$

$$\underset{0}{\underset{\parallel}{h^0(-E-E')}} - h^1(-E-E') + h^0(\underset{2E}{\underset{\wr}{\underline{K_X + E + E'}}}) = \chi(\mathcal{O}_X) + \tfrac{1}{2}(-E-E').(-E-E'-K_X) = 1 \qquad \because 2K_X \sim 0$$

$$\therefore \quad h^0(2E) = 1 + h^1(-E-E')$$

$\mathcal{O}_E \quad \because E' \cap E = \phi$

$\quad\quad\quad\quad \wr\!\wr$

$$0 \to \mathcal{O}_X(-E-E') \to \mathcal{O}_X(-E') \to \mathcal{O}_E(-E') \to 0$$

$$0 = H^0(X, \mathcal{O}_X(-E')) \to H^0(E, \mathcal{O}_E(-E')) \hookrightarrow H^1(X, \mathcal{O}_X(-E-E')) \Rightarrow h^1(-E-E') \geq 1$$

E' effective $\qquad H^0(E, \mathcal{O}_E(-E')) \cong \mathbb{C}$

$\therefore h^0(2E) \geq 2$

from previous discussion

$h^0(2E) = 2$

② $\underline{h^0(E, \mathcal{O}_E(E)) \neq 0}$, then $\mathcal{O}_E(E) \simeq \mathcal{O}_E$ & $h^0(E) = 2$

adjunction $\Rightarrow E^2 = 0 \quad \therefore |E|$ has no base points

$X \xrightarrow{|E|} \mathbb{P}^1$ elliptic fibration

$$\begin{array}{ccc} \pi^{-1}(E) \subseteq \tilde{X} & \xrightarrow[2:1]{\pi} & X \supseteq E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

$\tilde{X}$: K3

2 disjoint torus

Thus, π is a double cover

Möbius transf.

has two ramification points

$\Rightarrow$ The pre-images are the two multiple fibres E_1, E_2

$$2E_1 \sim E \sim 2E_2$$

$$\tilde{X} \xrightarrow{2:1} X \Rightarrow \pi_1(X) \simeq \mathbb{Z}_2 \quad \therefore H_1(X, \mathbb{Z}) \simeq \mathbb{Z}_2$$

$\pi_1(\tilde{X}) = 0$

universal coefficient theorem

$$0 \to \mathrm{Ext}^1(H_1(X,\mathbb{Z}), \mathbb{Z}) \to H^2(X,\mathbb{Z}) \to \mathrm{Hom}(H_2(X,\mathbb{Z}), \mathbb{Z}) \to 0$$

$$\Longrightarrow H^2(X,\mathbb{Z})_{tor} \cong \mathbb{Z}_2$$

generated by K_X, $2K_X \sim 0$

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\cong} & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

$$\Longrightarrow \quad K_X \sim E_1 - E_2 \qquad 2(E_1 - E_2) \sim 0$$

Proposition 3: Every Enrique surface admits an elliptic fibration.

pf: Let X be an Enrique surface.
$\tilde{X}$ be the double cover K3 surface.

Choose $d \in H^2(X,\mathbb{Z})/tor$ s.t $d^2 = 0$,

$$\begin{array}{c} H^2(X,\mathbb{Z})/tor \\ \cong \\ H \oplus (-E_8) \end{array}$$

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}_X) & \longrightarrow & \operatorname{Pic} X \cong & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ \| & & \cup & \cup & & \| \\ 0 & & \exists D & \longmapsto d & & 0 \end{array}$$

R.R $\quad \underbrace{h^0(X,D) + h^0(X,K_X-D)}_{\text{exactly one of them nonzero}} \geq \underset{\substack{\| \\ 1}}{\chi(\mathcal{O}_X)} + \frac{1}{2}D.(D-K_X) = 1 + \frac{1}{2}D^2 = 1$

Similarly, $\overbrace{h^0(X,-D) + h^0(X,K_X+D)} \geq 1 \qquad (K_X+D)^2 = D^2 = 0$

$|K_X \pm D|$ can't be both non-trivial since $(K_X+D) + (K_X-D) \sim 0$

Therefore, exactly one of $|D|, |-D|$ is non-empty.

In particular, $\exists$ effective D on X w/ $D^2 = 0$.

Similar to the K3 case, after finitely many Picard-Lefschetz reflection, we may assume that $D.C \geqslant 0$, $\forall$ (-2)-curve.

Claim: $D = m\mathcal{F}$. for some elliptic configuration.

Then the proposition follows from the Proposition 3.